

# The Lense–Thirring Effect and Mach’s Principle

Hermann Bondi<sup>†</sup> and Joseph Samuel

Raman Research Institute, Bangalore 560 080, India

## Abstract

We respond to a recent paper by Rindler on the “Anti–Machian” nature of the Lense–Thirring effect. We remark that his conclusion depends crucially on the particular formulation of Mach’s principle used.

<sup>†</sup> Permanent address: Churchill College, Cambridge, CB3 0DS, England

# 1 Introduction

In a recent paper, Rindler [1] has analysed the Lense–Thirring effect [2, 3, 4] and concluded that the result is anti-Machian. Rindler uses a particular interpretation of Mach’s principle. We wish to stress here that Rindler’s interpretation is only one amongst many. Indeed, the literature on this topic is so diffuse that we think it desirable to set out a list of interpretations that come to mind. Our list is far from exhaustive, but it is long enough to make numbering different versions necessary.

We begin with Mach0, which is the basis of the whole idea: *The universe, as represented by the average motion of distant galaxies [5] does not appear to rotate relative to local inertial frames.*

We illustrate this point by a modern version of Newton’s famous bucket experiment: the Sagnac effect. This effect provides an operational method for an observer to decide, by local measurements, if she is rotating. Consider an astronaut in an enclosed spaceship with angular velocity  $\omega$ . The astronaut takes a closed circular fibre optic tube at rest with respect to the spaceship and sends two rays of monochromatic laser light in opposite directions around the tube. These rays are made to interfere [6] after each ray has gone round once. If the spaceship is rotating, the corotating ray will take longer to come around than the counter-rotating one, leading to an arrival time difference, which can be observed as a fringe shift. The time difference is given by:  $\Delta t = -4A\omega/c^2$ , where  $\omega$  is the angular velocity of the spaceship and  $A$  is the area enclosed by the tube. Using the Sagnac effect, one can by experiments internal to the spaceship, so arrange the angular velocity of the spaceship that the Sagnac shift (defined as  $\Delta t/2$ ) vanishes. A frame at rest with respect to such a spaceship is called a locally non rotating frame. Sit in this frame, look

up at the sky and note that the distant galaxies are still. Mach's principle (Mach0) is the experimental observation that the inertial frame defined by local physics (zero Sagnac shift) coincides with the frame in which the distant objects are at rest.

Mach0 is an experimental observation and not a principle. One could interpret Mach's writings as a suggestion to construct a theory in which Mach0 appears as a natural consequence. But Mach's writings have been variously interpreted. Our purpose here is to list a number of interpretations of Mach's principle and view them in the light of currently accepted theories in an effort to refine and clarify the idea.

We do have at our disposal two well established theories of space, time, gravity, matter and motion –Newton's and Einstein's– both experimentally successful in their respective domains of validity. Newton's holds that space and time are absolute. Einstein's holds that space time geometry is affected by matter. There is no question (as Rindler observes) that these experimentally successful theories are here to stay regardless of whether they satisfy any of the rather philosophical criteria embodied in Mach's principle.

## **2 Versions of Mach's Principle**

Recent discussions of Mach's Principle, including this one, have greatly benefited from the 1993 Conference organised by J. Barbour and H. Pfister and the excellent book [7] resulting from it. A glance at the book (note especially J. Barbour's list on page 530) will show that there have been numerous interpretations of Mach's writings. For an authoritative account of the history of Machian ideas, the reader is referred to [7].

We now list a few versions of Mach's principle which appear in the lit-

erature. Each statement of Mach’s principle, will be accompanied by a declaration of the theoretical framework in which it is intended to apply. Two levels of compatibility will be considered: Does the particular statement of Mach’s Principle make sense in the theory, and secondly, is it satisfied by it? We use the letters N and E to refer to Newtonian and Einsteinian space time. Even within Einstein’s theory there is a further dichotomy– is one discussing cosmology (the whole universe) or an isolated system embedded in an asymptotically flat space time? This distinction is made by the notation EA for asymptotically flat spacetimes and EC for relativistic Cosmologies. Our purpose in compiling this list is to draw attention to the diversity of ideas that pass under the guise of “Mach’s principle”. (Page numbers refer to [7] unless otherwise indicated.)

- Mach1: *Newton’s gravitational constant  $G$  is a dynamical field.* (Makes sense in N, EA, EC.) Mach1 is not true in N or E. This version applied to Einstein’s theory has led to Brans–Dicke Theory[8, 9].
- Mach2: *An isolated body in otherwise empty space has no inertia (pp 11,39,181, 185).* (Makes sense in N, EA, EC.) Neither Newtonian nor Einsteinian gravity satisfy this version. In both theories the motion of an isolated body is determined and not arbitrary.
- Mach3: *local inertial frames are affected by the cosmic motion and distribution of matter (p92).* (Makes sense in N, EA, EC [10] .) This version is closest to the bucket experiment. In this form, Newton’s theory is in clear conflict with Mach3. Einstein’s theory is not (see section 4 below).
- Mach4: *The universe is spatially Closed (p 79).* (Makes sense only in

EC.) We do not know if Mach4 is true.

- Mach5: *the total energy, angular and linear momentum of the universe are zero (p237).*(Makes sense in N, EA, EC.) It is not true in N and EA. In EC it is claimed [11] that the total angular momentum of a closed universe must vanish.
- Mach6: *Inertial mass is affected by the global distribution of matter (pp 91,249).* Makes sense in (N, EA, EC). Is not true in any of them. Hoyle and Narlikar [12] proposed a theory in which implements Mach6.
- Mach7: *If you take away all matter, there is no more space [13].* Makes sense in (N,EA,EC). Not true in any of them.
- Mach8:  $\Omega = 4\pi\rho GT^2$  *is a definite number of order unity (p475).* (Here,  $\rho$  is the mean density matter in the universe and  $T$  is the Hubble time. Makes sense in EC only.)  $\Omega$  does seem to be of order unity in our present universe, but note that of all EC models, only the Einstein–DeSitter makes this number a constant, if  $\Omega$  is not *exactly* one. Making a theory in which this approximate equality appears natural is a worthwhile and ongoing effort (eg inflationary cosmologies).
- Mach9: *The theory contains no absolute elements ([14].* (Makes sense in N, EA and EC) This version is clearly explained by Jürgen Ehlers in [7] p 458. The elements (fields, for example) appearing in the theory can be divided into dynamical (those that are varied in an Action principle) and absolute (those that are not). The Action principle leads to equations for the dynamical fields to satisfy. The absolute elements are predetermined and unaffected by the dynamics.

Newton’s theory does not satisfy Mach9 (space and time are absolute) and neither does EA (asymptotic flatness introduces an absolute element—the flat metric at infinity). EC does satisfy Mach9 [15]. From the point of view of invariance groups (J.L Anderson, A. Trautmann, quoted on p 468 [7]) Mach9 is the requirement that the invariance group of the theory is the *entire* diffeomorphism group of spacetime. Viewed in this light Mach9 is just the principle of general covariance.

- Mach10: *Overall rigid rotations and translations of a system are unobservable.* (This version makes sense *only* in N; In Einsteinian spacetime one has no idea what a rigid rotation is anymore than one knows what a rigid body is.) This is *not* satisfied in Newtonian theory. If one insists on the principle and constructs a theory which satisfies it, one is led [16] to a class of models (called “relational” by Barbour and Bertotti [16]). There is considerable literature on these models [7, 17]. We spend a few words on these models and their connection with Newtonian theory.

*Relational Models:* Let  $x_a^i$ ,  $i = 1, 2, 3$ ,  $a = 1 \dots N$  be the positions of  $N$  particles in Newtonian spacetime and  $p_{ia}$  their conjugate momenta. The Hamiltonian  $H(x, p)$  determines the time evolution of  $(x_a^i, p_{ia})$  via Hamilton’s equations. The transformation

$$\begin{aligned} x_a^i(t) &\rightarrow R_j^i(t)x_a^j(t) \\ p_{ia}(t) &\rightarrow R_i^j(t)p_{ja}(t), \end{aligned} \tag{1}$$

where  $R_j^i(t)$  is an arbitrary time dependent rotation matrix maintains the distance relations between the  $N$  particles. If a model is relational [16], such a transformation is unobservable, like a “gauge transformation” in electrodynamics. From Dirac’s theory of constrained systems

[18, 19], it follows that the transformations (1) must be generated by first class constraints. The generator of overall rotations of the system is the total angular momentum:

$$J^i = \Sigma_a \epsilon^{ijk} x_{aj} p_{ak}.$$

Thus the system is subject to the constraints

$$\phi^i(x, p) := J^i - C^i \approx 0,$$

where  $C^i$  are constants. The requirement that the constraints be first class in the sense of Dirac [18] forces the constants  $C^i$  to vanish.

The extended Hamiltonian in the sense of Dirac is

$$H_E(x, p) = H(x, p) + \omega_i J^i,$$

where  $\omega^i$  are arbitrary functions. While we have only dealt with overall rotations in (1), one can similarly deal with arbitrary translations and arbitrary time reparametrizations. Relational models can be thus derived from Newtonian Hamiltonian mechanics by imposing constraints on the phase space so that the total angular momentum, momentum and Energy vanish.

These relational models are clearly distinct from Newtonian theory. For instance, Newtonian theory admits solutions with nonzero angular momentum (like the solar system in an otherwise empty universe) while relational models do not permit such solutions.

### 3 Rindler's Criticism

We now briefly summarise Rindler's argument. Consider the earth in an otherwise empty universe. Let  $O$  be a reference frame rigidly attached

to the earth. Suppose that a gyroscope  $G$  is taken around the earth in the equatorial plane along a circle of radius  $r$  with a constant clockwise angular velocity  $\Omega$ . To keep track of orientations, we suppose the earth and the gyroscope marked with cross hairs (as in Fig.1 of Rindler). We arrange that the orientation of  $G$  relative to the earth's is constant during the motion. (Rindler uses the Schwarzschild metric outside the earth to compute  $\alpha$  the precession rate of the gyroscope. We choose the radius  $r$  to set  $\alpha$  to zero. It simplifies the argument.)

Now view the situation from the point of view of an observer  $O'$ , who rotates rigidly relative to  $O$  with constant clockwise angular velocity  $\Omega$ .  $O'$  sees the earth rotating anticlockwise with angular velocity  $\Omega$ , the centre of the gyroscope at rest. Notice however, that the gyroscope (which was not rotating with respect to  $O$ ) now rotates anticlockwise with angular velocity  $\Omega$  relative to  $O'$ . Thus the gyroscope rotates *in the same sense* as the earth.

It follows from Mach10 that a rotating body in otherwise empty space makes the local compass of inertia take up *all* of the body's angular velocity. Applied to the earth, which is not in empty space but in the universe, one would expect that the effect of the earth on the gyroscope should be considerably diluted by the effect of the rest of the universe. Thus one would expect that the local compass of inertia would take up a small *positive* fraction of the earth's angular velocity. The sign of this effect is everywhere positive unlike the sign of the Lense–Thirring effect. This is the basis for Rindler's conclusion that the Lense–Thirring effect is Anti–Machian.

## 4 The Lense-Thirring effect as Machian

We now show that one can arrive at the opposite conclusion from Rindler's by using a different version of Mach's Principle. We use the often employed exact analogy between rotation in General Relativity and magnetic fields [20] to deduce that the slight influence of a spinning body on the rotation of the near-by compass of inertia goes with that of the body near the poles and in the opposite sense in the equatorial plane.

*The Lense-Thirring effect:* Consider a stationary spacetime *i.e* one with a timelike Killing vector  $\xi: \nabla_a \xi_b + \nabla_b \xi_a = 0$ . One can adapt the time coordinate to  $\xi$  so that  $\xi = \partial/\partial t$  and the metric assumes the form:

$$ds^2 = g_{00}(dt + A_i dx^i)^2 - \gamma_{ij} dx^i dx^j,$$

where  $A^i = g_{0i}/g_{00}$ . The coordinate transformations that preserve this form include  $t \rightarrow t + \alpha(x^i)$ , which physically represents the resetting of clocks. Under such transformations  $A_i$  transforms as  $A_i \rightarrow A_i + \nabla_i \alpha$  like the vector potential in electrodynamics. Consequently its curl  $F_{ij} := \partial_i A_j - \partial_j A_i$  is invariant and represents rotation of the spacetime (more geometrically, the failure of  $\xi$  to be hypersurface orthogonal). It is easily seen that a stationary Sagnac tube will measure a Sagnac shift of  $\oint A_i dx^i$ . A locally nonrotating Sagnac tube (one that measures zero Sagnac shift) will appear to rotate as viewed from infinity. The angular velocity of rotation has the spatial distribution of a dipole magnetic field and *reverses sign between the equator and the poles*. As we show below this is *exactly* what one expects from Mach's principle (Mach3).

If one applies Mach's Principle in the form Mach3 to understanding rotation in General Relativity, one sees that the prediction of Mach3 agrees

with the sense of the Lense–Thirring effect. If one is stationary at the north pole of the earth one sees the earth rotating anticlockwise and one also sees a “non rotating” gyroscope (one which registers a null Sagnac effect) rotating anticlockwise. (The magnitude of the effect is not in question here only its sign.) The agreement between the sense of the Lense–Thirring effect and Mach3 also extends to the equatorial plane. It is true that the sense of the Lense–Thirring effect reverses at the equator. It is also true that the prediction of Mach3 reverses: An observer in the equatorial plane sees the *nearer parts* of the Earth moving past her sky in an *clockwise* direction around the North Star. While the *further parts* are moving in an *anticlockwise* direction, the sense of the effect is dominated by the nearer mass. The net effect is as *clockwise* rotation of a locally nonrotating gyroscope. Thus Mach3 agrees with the Lense–Thirring effect both at the poles and the equator. (For a somewhat different argument leading to the same result, see the articles by Schiff and Thorne, quoted on page 321 of [7]).

## 5 Conclusion

The list given above shows the variety of interpretations that Mach’s writings have spawned. Some of them express the idea “Cosmic conditions affect local physics”. Others state requirements to be satisfied by physical theories. There are also logical relations between some of the versions: for instance Mach10 (which is formulated in N) implies that the total angular momentum, momentum and energy of the Universe is zero. This is precisely the content of Mach5, which is formulated more generally. On the other hand, Mach1 has no obvious connection with Mach0.

To us, the most remarkable feature of the list (which Rindler’s paper [1]

brings to light) is that two entries in it (Mach3 and Mach10) give rise to *diametrically opposite* predictions, when applied to a simple physical situation. By popular usage Mach's principle has acquired a range of meanings, some of which are in conflict with each other. Mach's writings have been a source of inspiration to many (including Einstein). We hope that our effort at distinguishing between existing versions of Mach's Principle will serve to clarify ideas and eliminate needless controversy.

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- [5] Though the early authors spoke of the "fixed" stars, their movement in our rotating Galaxy is indeed observable with modern instrumentation.
- [6] The apparatus described here is a laser ring gyro, which is routinely used in inertial navigational systems in aircraft. With modern technology [21], the minimum detectable rotation rate is .2 degrees/hour (using an integration time of 30 seconds). For recent developments in this area

- see the references given in Ke-Xun Sun et al Phys. Rev. Lett. **76**, 3053 (1996).
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